Structure of Lift-Generated Rolled-Up Vortices

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Widely used analytical models describing the structure of lift-generated vortices behind aircraft wings (e.g., models of Betz or Lamb) are generally not in acceptable agreement with experiments. The lack in conservation of vortical dispersion and rotational energy, which is connected with the induced drag of the generating wing, is considered as a reason for this discrepancy.

The model of Lamb is modified in a simple way to account for the conservation laws. These modifications yield a reduction in the core radius and a remarkable increase in the maximum tangential speed. A comparison with experiments is made and reasonable agreement is found. For the modification of the vorticity distribution in the Lamb model, the Kirchhoff-Routh function is used. This function is proportional to the rotational energy for a continuously spaced array of point vortices having the proper vorticity. The proposed model can be adapted to arbitrary vorticity distributions and, hence, wing loadings.

Nomenclature

a	= parameter of vortical dispersion [Eq. (27)], m
b	= wing span, m
c	= parameter in the modified Lamb model [Eq. (29)], 1/m
c_R	= vortex structure parameter [Eq. (5)]
D_e	= induced drag at elliptic loading, N
D_i	= induced drag, N
I	= second moment of vorticity, m^4/s
ι	= radius of vortical dispersion ($=\sqrt{I/\Gamma_0}$), m
k_ι	= circulation around a point vortex, m^2/s
p	= static pressure, N/m^2
p_o	= freestream static pressure, N/m^2
q	= part of circulation in the inner region of the vortex
r	= vortex radial coordinate, m
r_C	= radius of vortex core, m
r_K	= radius of vortex, m
s	= wing semispan, m
s'	= tip vortex semispan, m
t	= time since wake generation, s
U	= freestream velocity = airspeed, m/s
u	= axial velocity, m/s
v	= tangential velocity, m/s
v_m	= maximum tangential velocity, m/s
w	= radial velocity, m/s
W	= Kirchhoff-Routh function [Eq. (28)], m^4/s^2
Γ_0	= wing root circulation, m^2/s
γ_r, γ_η	= normalized distribution of circulation
δ	$= \epsilon_R / D_e$
ϵ_R	= rotational kinetic energy/unit length, N
η	$= y/s =$ dimensionless distance in spanwise direction
λ	= parameter in the Lamb model [Eq. (21)], $1/m^2$
ϱ	= air density, kg/m^3

Introduction

IT is well known that a finite-span wing sheds a vortex sheet that rolls up downstream of the wing to form two oppositely rotating tip vortices. The flow in these trailing vortices poses potential hazards to lighter aircraft which follow or cross the path of a heavier aircraft.

A lot of experimental studies have been conducted to determine velocity fields, decay, and motion of trailing vortices.[1-4] These investigations cover flight tests as well as experiments in wind tunnels and water towing tanks. One of the main concerns is the measurement of the tangential speed v as a measure of the potential hazard produced by the wake vortices.

Earlier theoretical treatments of the rolling up of the trailing vortex sheet are generally based on the papers of Kaden, Betz and Spreiter/Sacks.[5-7] The method of Kaden[5] and Betz[6] is a two-dimensional, inviscid analysis of the rollup process. It leads to a model of the vortex structure which conserves the first and second moment of vorticity. This theory must fail near the core of the vortex where viscosity and turbulent shear are of overriding importance. Therefore, the Betz model[6] does not accurately describe the distribution of the tangential speed v near the core radius; since this is the position of maximum v, it provides little help for the correlation with the mentioned measurements.

As more realistic descriptions of the vortex structure, the models of Rankine and Lamb[8] have been used. In the Rankine vortex model the vorticity is concentrated inside the vortex core with a homogeneous distribution. Lamb[8] found a Gaussian vorticity distribution through a similarity solution of the Navier-Stokes equations for a two-dimensional, axisymmetric flow in an incompressible fluid. A number of researchers have proposed that the structure and the decay of tip vortices should follow the general form of the Lamb model,[8] but with the kinematic viscosity term replaced by an empirically determined eddy viscosity which accounts for the turbulence in the vortex core.[9,10]

Spreiter and Sacks[7] have introduced another relationship for the behavior of the rolled-up vortices far behind the wing. Assuming a Rankine vortex model, they calculated the kinetic energy of the fluid (per unit length in the stream direction) outside and inside the vortex core. By equating the kinetic energy per unit length to the induced drag of the wing, Spreiter and Sacks obtained an expression relating the radius of the core, r_C, to the induced drag[7]

$$D_i = \frac{\rho \Gamma_0^2}{2\pi}\left[0.25 + \ln \frac{2 - r_C/s}{r_C/s'}\right] \tag{1}$$

The terms in the bracket are related to the energy inside and outside the vortex core, respectively.† The semispan of the

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†In this case of two-dimensional flow, the contribution of the axial velocity component to the kinetic energy is neglected (see Sec. II).

vortices, s', depends on the lift distribution of the wing (semispan s) If the span loading is elliptical, s' and r_C are given by

$$s' = s\pi/4 \qquad r_C = 0.154\ s \tag{2}$$

In principle, the kinetic energy of any vortex pair can be calculated if the vortex structure is given. Then, relationship (1) can be generalized and yields an additional condition which must be fulfilled by vortex models

The generally accepted need for basic research on vortex dominated flows has also prompted a search for efficient basic computational methods for prediction of vorticity fields Numerous approaches, focused on the rollup of vortex wake behind wings, have been tried[11,12] or are under development At present, none of these methods can accurately describe the structure of a rolled up vortex near the core Comparisons with experimental data, provided it is possible at all, show only poor agreement

This paper suggests a simple model of the tip vortices which takes into account both the laws of conservation of vortical dispersion (as in Kaden[5] and Betz[6]) and of energy (as in Spreiter and Sacks[7]) It is argued that this combination of conservation laws gives a better agreement with the experiments Such a simple model seems to have some benefit even in the age of numerical methods since it gives a simple, rapid estimation of the concentration of vorticity This estimate can be used in developing the requirements for more complex computational codes. Also, the model is flexible and tip vortices generated from arbitrary spanload distributions can be described

In the context of this paper, an earlier contribution by Birkhoff and Fisher[13] on the concentration of vorticity is worth mentioning. Even if the author does not agree with some of the conclusions drawn in Ref. 13, one should note that Birkhoff and Fisher have stated the importance of the invariant energy for describing the rollup of vortex sheets

Kinetic Energy of a Vortex Pair

It is assumed that the vorticity in each vortex of a pair is distributed in an arbitrary but axisymmetric form, so that the circulation can be expressed as

$$\Gamma = \Gamma_0 f(r)$$

The function $f(r)$ will approach a value of 1 for large distances compared to the radius of the vortex, and $f(r)$ may contain parameters which must be determined experimentally or can be fixed by theoretical arguments The influence of time is neglected since the aging process of a rolled up vortex is not considered herein

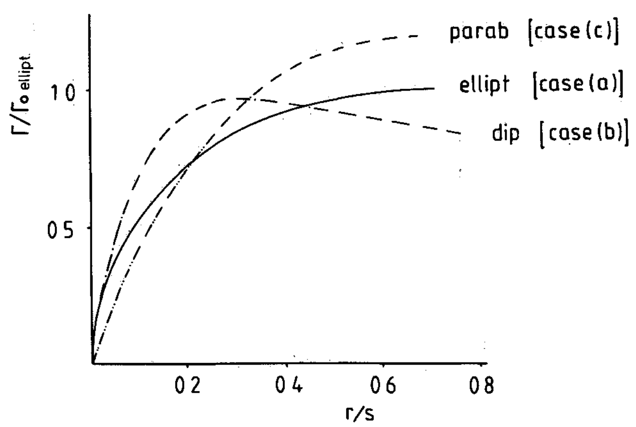

Fig 1 Distribution of circulation in a rolled up vortex for three types of spanwise loading

First we divide each vortex into an inner and outer region, separated by the radius r_K, such that for the outer part $\Gamma > 0.99\Gamma_0$ Then, r_K must obey the equation

$$f(r_K) = 0.99$$

The "rotational" energy of the inner part of a single vortex (per unit length) is given by

$$\frac{\rho}{2}\int_A (v^2 + w^2)\,dA = \frac{\rho\Gamma_0^2}{4\pi}\int_0^{r_K}\left(\frac{\Gamma}{\Gamma_0}\right)^2 \frac{1}{r}\,dr \tag{3}$$

where v and w are orthogonal components of the velocity in the lateral plane and A the end face of a cylinder with radius r_K

The total kinetic energy of the vortex pair (per unit length) can be approximately written in the form (see Ref 7)

$$\epsilon_R = \frac{\rho\Gamma_0^2}{2\pi}\left[c_R + \ln\frac{2 - r_K/s'}{r_K/s'}\right] \tag{4}$$

where c_R is given by the integral in Eq. (3)

$$c_R = \int_0^{r_K}\left[\frac{\Gamma}{\Gamma_0}\right]^2 \frac{1}{r}\,dr \tag{5}$$

Using the momentum integral theorem,[14] the drag on a wing is given by

$$D = \int_A \left[(p_0 - p) + \rho u(U - u)\right] dA$$

where u and p are the axial component of the velocity and the pressure at the downstream end face A of the control surface, and U and p_0 the corresponding values at the upstream end face. If the total head loss is neglected, D is reduced to the induced drag D_i and can be written as

$$D_i = \frac{\rho}{2}\int_A \left[(v^2 + w^2) - (U - u)^2\right] dA \tag{6}$$

This equation shows that the rotational energy (per unit length) must be slightly larger than the value of the induced drag

At the beginning of the aging process of the tip vortex, the contribution of $(U-u)^2$ to the integral is limited to the small cross section of the vortex core so that

$$\epsilon_R = D_i \tag{7}$$

is a good approximation

For a given vortex structure Eqs (4) and (5) yield the kinetic energy, which can be compared with the condition defined in Eq (7) Equations (4) and (7) can be used to fix parameters which may be contained in $f(r)$

Table 1 Relationship between energy conservation and vortex size ($c_R = 1$)

δ	r_K/s
1 2	0 148
1 1	0 185
1 0	0 229
0 9	0 282
0.8	0.343
0.7	0.414

For an elliptical lift distribution, D_i is related to Γ_0 by

$$D_i = D_e = \frac{\pi}{8}\rho\Gamma_0^2 \tag{8}$$

Then, Eqs. (4) and (8) lead to

$$\frac{r_K}{s} = \frac{r_K}{s'}\frac{\pi}{4} = \frac{\pi/2}{1+\exp(\delta\pi^2/4 - c_R)} \tag{9}$$

where $\delta = \epsilon_R / D_e$ is equal to 1 for an elliptic span loading if the energy is conserved during the rollup. Equation (9) is not very accurate but can be used for a survey. An evaluation of Eq. (9) reveals the importance of the conservation of rotational energy. The structure of the vortex is very sensitive to changes in δ, as can be seen in Table 1.

A more general procedure for the calculation of the kinetic energy of vortex fields is given in Sec. V, where nonaxisymmetric distributions of vorticity are treated.

Analysis of Vortex Models

Betz Model

The vortex model of Betz[6] is based on the assumption that the vortical dispersion of the plane vortex sheet of each wing half-span is retained during rollup. For an elliptical lift distribution the circulation of the wing is given by

$$\Gamma/\Gamma_0 = \sqrt{1-\eta^2} \qquad \eta = y/s \tag{10a}$$

The conservation of vortical dispersion requires a transformation of the spanwise coordinate η into the radial distance $r = r(\eta)$ given by[15]

$$\frac{r(\eta)}{s} = \frac{\pi/4 - 0.5\arcsin\eta}{\sqrt{1-\eta^2}} - \frac{\eta}{2} \tag{11}$$

In the Betz wake model,[6] the separation distance, 2 s', between the cores of the vortex pair is equal to the separation distance between the centroids of vorticity shed by the wing semispans. For the distribution, Eq. (10a), it follows that

$$s' = s\pi/4 \tag{12}$$

Combining Eqs. (10a) and (11) yields the circulation as well as the tangential speed as a function of r

As has been shown by Jordan,[16] the radial distribution of circulation in the rolled-up vortex can easily be determined for an arbitrary spanwise distribution of circulation over the wing. If $\Gamma_0\gamma_r(r)$ and $\Gamma_0\gamma_\eta(\eta)$ are the radial and spanwise distributions, r and η are related by the requirement

$$\gamma_\eta(\eta) = \gamma_r(r)$$

This establishes a function $r = r(\eta)$

$$\frac{r(\eta)}{s} = \frac{1}{\gamma_\eta(\eta)}\int_\eta^1 \gamma_\eta(z)\,\mathrm{d}z$$

The center of the rolled-up vortex is located at

$$\eta = \int_0^1 \gamma_\eta(z)\,\mathrm{d}z$$

In addition to elliptic loading [Eq. (10a)], two further distributions have been considered in Ref. 16.

$$\gamma_\eta(\eta) = 1 + 1.5\eta^2 - 2.5\eta^4 \tag{10b}$$

$$\gamma_\eta(\eta) = 1 - \eta^2 \tag{10c}$$

Distribution (10b) has a slight central dip, while Eq. (10c) represents a parabolic distribution. The induced drag of both distributions is larger than the minimum value for elliptic loading, Eq. (10a), by the same factor of 9/8.

A first shortcoming of the Betz method is the fact that the tangential speed does not go to zero at the center of the vortex core. Another deficiency involves the rotational energy, ϵ_R. Moore and Saffman[17] have already stated that for an elliptic loading "the final roll-up state given by Betz does not conserve energy." This shortcoming of the Betz method is not limited to elliptic loading. The evaluation of the energy for Eqs. (10a-c) leads to the following results

$$\epsilon_{R(a)} = 0.93$$

$$\epsilon_{R(b)} = 0.89$$

$$\epsilon_{R(c)} = 0.96$$

Therefore, the statement of Moore and Saffman[17] can be generalized, that the Betz method gives rolled-up vortex states with values of the rotational energy which are—depending on the spanwise loading—too low. These results, together with those in Table 1, indicate that the inviscid Betz concept cannot accurately describe the structure of tip vortices.

Rankine Vortex

The Rankine vortex is a simple model with just one parameter: the radius of the vortex core, r_C. The circulation is given by

$$\Gamma/\Gamma_0 = (r/r_C)^2 \qquad (r<r_C)$$
$$= 1 \qquad (r>r_C) \tag{13}$$

As already mentioned, the energy condition in Eq. (7) can be satisfied by a value of

$$r_C/s = 0.154 \qquad \text{elliptical loading} \tag{14}$$

The maximum of the tangential speed can be found from Eqs. (13) and (14) as

$$\frac{v_m s}{\Gamma_0} = \frac{1}{2\pi r_C/s} = 1.03 \tag{15}$$

a value which does not show an acceptable agreement with experiments (see Sec. IV).

While the Rankine vortex fulfills the energy condition in Eq. (7), it fails to conserve the second moment of the vorticity shed by each half-span of the wing. The dispersion of the vorticity can be written in the form

$$I = \Gamma_0 i^2 = \int_0^{r_K} \Gamma_0 r^2 \frac{\mathrm{d}(\Gamma/\Gamma_0)}{\mathrm{d}r}\mathrm{d}r \tag{16}$$

For an elliptical loading, the square root average radius over which the circulation is spread is given by

$$i/s = 0.223 \tag{17}$$

This value should be retained in the concentrated trailing vortex, even if we do not use the stronger assumption of Betz that the vortical dispersion of every part of the vortex sheet, convoluted into a rolled-up form, is retained.

The vortical dispersion of a "concentrated" tip vortex with axisymmetric structure can be obtained from

$$i=\sqrt{\int_0^{r_K} r^2 \frac{\mathrm{d}(\Gamma/\Gamma_0)}{\mathrm{d}r}} \tag{18}$$

Using the relation in Eq. (13) we obtain for a Rankine vortex,

$$i/s=(1/\sqrt{2})r_C/s \tag{19}$$

Combining Eqs. (14) and (19) we obtain

$$i/s=0.109 \tag{20}$$

a value which is much too low compared with the required 0.233.

The relation in Eq. (19) implies that an increase of i/s would lead to a larger core radius and, therefore, to a reduction of the kinetic energy. This is incorrect. Thus, the Rankine vortex is an unsuitable model for describing the structure of a tip vortex.

Lamb Vortex

The Lamb model has been used quite often for correlating measurements. The circulation of a single Lamb vortex is given by

$$\Gamma=\Gamma_0(1-e^{-\lambda r^2}) \tag{21}$$

where the parameter λ can be found by experiment or by theoretical considerations. In general, λ is a function of time, and describes the aging process. The condition $\epsilon_R=D_e$ leads to the following values:

$$c_R=0.707, \qquad \lambda=87/s^2, \qquad r_K/s=0.230, \tag{22}$$

$$r_C/S=0.120, \qquad \frac{v_m s}{\Gamma_0}=0.95$$

Applying Eq. (18) to the Lamb vortex, the radius of inertia of the vorticity is found as $i/s=0.104$. This value is close to that of the Rankine vortex and again much too low.

New Simple Vortex Model

The vortex models discussed thus far have shown some principal shortcomings, which make them unsuitable for accurately predicting maximum tangential speed. This is illustrated in Fig. 2 where several measurements of the tangential speed in tip vortices are reproduced. While the Betz model fails completely, the Rankine and Lamb models considerably underestimate the tangential speed. The parameters in these models are chosen in such a way that the vortex energy is conserved. There is some hope that a model which meets the conditions of both Eqs. (7) and (17) will show a better agreement with experiment.

It is not easy to satisfy both conditions simultaneously. In order to increase the vortical dispersion, vorticity must be brought away from the vortex core. This will, in general, reduce vortex energy. Therefore, it is necessary to redistribute the vorticity in such a way that one part is concentrated nearer to the vortex center while another smaller part is brought farther away from the vortex core. This can be done in a number of ways. An easy way is the following adaptation of the Lamb model: A Lamb vortex with the core radius r_C and the reduced circulation $q\Gamma_0(q<1)$ is positioned in the center part. The remaining part of the vorticity is placed at a larger radius r_K, bringing the circulation around the vortex to the full value of Γ_0. This modification of the Lamb vortex leads to a model which contains three parameters (instead of one in the basic model): r_C, r_K, and q. For a given value of any one of these parameters, the values of r_C/s and r_K/s can be obtained by applying the conservation laws. It is not obvious, however, that this simple model will satisfy the conditions in Eqs. (7) and (17) within a reasonable range of parameters.

Using the above distribution of vorticity, the integral in Eq. (5) leads to

$$c_R=q^2(0.25+\ln r_K/r_C) \tag{23}$$

Thus, the energy condition defined by Eqs. (7) and (9) yields for elliptic loading

$$\frac{r_C}{s}=\frac{\pi/2}{(r_K/r_C)[1+(r_K/r_C)^{-q^2}\exp((\pi^2-q^2)/4)]} \tag{24}$$

A second relationship, which is equivalent to the one in Eq. (19), can be obtained by inserting the assumed vorticity distribution of the modified vortex model into Eq. (18).

$$\frac{r_C}{s}=\frac{i/s}{\sqrt{q/2+(1-q)(r_K/r_C)^2}} \tag{25}$$

Elimination of r_C/s by combining Eqs. (24) and (25) gives a relation between r_K/r_C and q, which can be solved iteratively to yield r_K/r_C as a function of q. Then, r_C/s follows from Eq. (24) or Eq. (25) and the maximum tangential speed at $r=r_C$ is found to be

$$\frac{v_m s}{\Gamma_0}=\frac{q}{2\pi}\frac{1}{r_C/s} \tag{26}$$

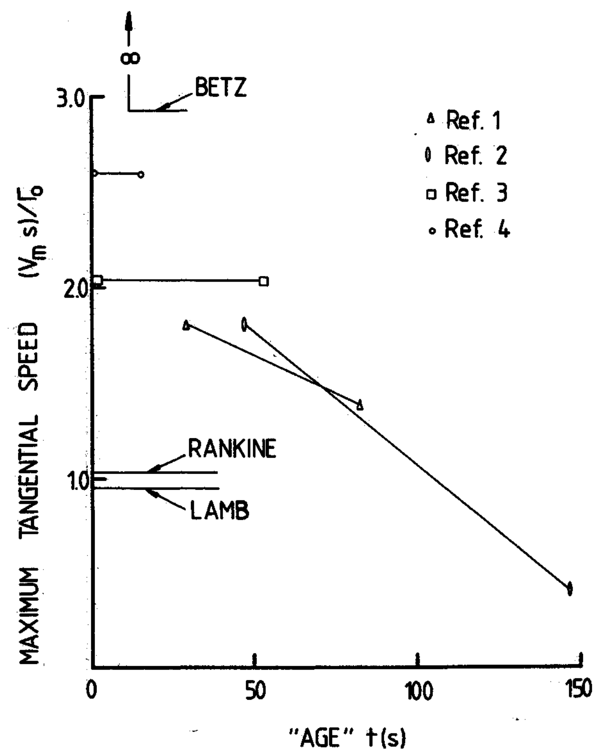

Fig. 2 Measured maximum tangential speed in rolled-up vortices.

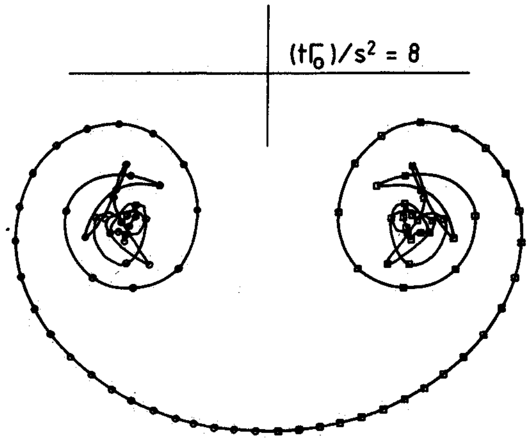

Fig 3 Rollup of a full span vortex sheet approximated by point vortices

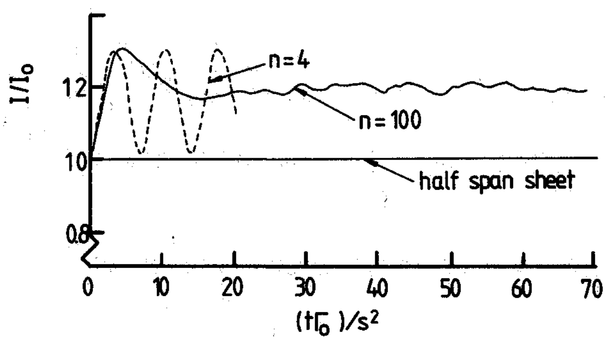

Fig. 4 Vortical dispersion during the rolling up of a vortex sheet represented by 4 and 100 point vortices

Table 2 Effect of varying q on vortex structure

q	r_C/s	r_K/s	$v_m s/\Gamma_0$
0.90	0.0565	0 695	2 54
0 85	0 0526	0 569	2 57
0 80	0.0453	0 495	2.81
0 75	0 0372	0.444	3 21
0 70	0 0291	0.406	3.82

Sample calculations for a few values of q using Eqs (24-26) are reproduced in Table 2

It can be seen that the modified vortex model leads to core radii considerably smaller than those obtained from the Rankine and the Lamb vortices together with higher value of the maximum tangential speed The results for $v_m s/\Gamma_0$ in Table 2 are in much better agreement with the experimental results in Fig 2. Best results are obtained for $0\ 8 < q < 0\ 9$ It should be emphasized that for these results the vorticity of one vortex does not penetrate the region of the opposite vortex ($r_K/s < s'/s = \pi/4$) so that the basic assumption is not violated

We may conclude that it is worthwhile constructing vortex models which satisfy Eqs (7) and (17) Even with the rather crude assumption made for the vorticity distribution, encouraging results were obtained It is, of course, possible to refine this model. However, even a refined model would be empirical and in some way arbitrary To further improve accuracy it would seem to be necessary to use a more sophisticated computational method Accordingly, a numerical study of two-dimensional vortex motion using point vortices is described in the next section.

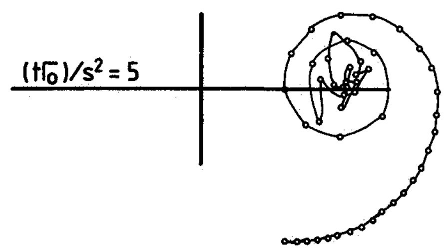

Fig 5 Roll up of a half-span vortex sheet approximated by point vortices

Discrete Vortex Approximation

As is well known, vortex motion at high Reynolds numbers can be studied by replacing the rotational region, e.g , the vortex sheet, by a number of discrete vortex filaments Their motion can be followed in time by calculating the convection field using the law of Biot-Savart Rosenhead[18] was the first to apply this method; subsequently, Westwater[19] carried out a calculation to study the rollup of a vortex sheet using the method Kuwahara and Takami[20] and Williams[21] proposed an improved method by superimposing elementary viscous vortex models of the Lamb type, each component having its own circulation k_j and a vortical dispersion that depends on time By introducing a high value of eddy viscosity for each individual vortex, the growth of the strong "irregularities" which arise in the region of high concentration of vortex filaments can be avoided.

Since we have seen that conservation of vortical dispersion and rotational energy can have an important influence on the prediction of simple models, we examine whether the point vortex model also conserves these quantities Since we are again not interested in the aging processes, we eliminate the influence of time by choosing a fixed vortical dispersion for each vortex filament Each discrete vortex is subjected to the velocity field induced by all the other vortices Thus, the equations of motion of the ith vortex are given by

$$\frac{dx_i}{dt} = -\frac{1}{2\pi}\sum_{i\neq j} k_j\left[1-\exp(r_{ij}/a)^2\right]\frac{y_i-y_j}{r_{ij}^2} \qquad (27)$$

$$\frac{dy_j}{dt} = \frac{1}{2\pi}\sum_{i\neq j} k_j\left[1-\exp(r_{ij}/a)^2\right]\frac{x_i-x_j}{r_{ij}^2}$$

where

$$\sqrt{r_{ij}} = (x_i-x_j)^2+(y_i-y_j)^2$$

We first consider the vortical dispersion, Eq. (16), during the rollup process. Again, an elliptical loading is assumed Figure 3 presents a result of numerical calculations of the rollup of a vortex sheet which is represented by 100 vortices These results are based on point vortices ($a=0$), and would be about the same for small values of a/s. Although large irregularities in the vortex motion can be observed, the vortical dispersion shows relatively small fluctuations (Fig 4) The calculations show that the Betz assumption, according to which the vortical dispersion of each half span is retained during the rollup, is only approximately valid The dispersion increases quite rapidly during the beginning of the rollup, and then fluctuates around an average value which is about 20% higher than the initial one, the Betz value.

To check whether this change in the dispersion is caused by the numerical method, we also computed the rollup of an isolated vortex sheet which only simulates the trailing vortices of one half-span (Fig 5) Again, the vortex motion is highly irregular but the vortical dispersion, also shown in Fig 4, does not fluctuate at all and retains its initial value. Thus, we can conclude that the violation of the Betz assumption is not due to the numerical method. Variation of vortical dispersion can also be obtained for the full span case if each half-span consists of only two trailing vortices, $n=4$. In this simple case the vortical dispersion increases as well and oscillates around an average value which is again 20% higher than the initial dispersion (Fig. 4). In the case of a larger number of vortices the oscillation is changed into a stochastic fluctuation with a smaller variance. Even if we consider the conservation of vortical dispersion of each half-span as an acceptable approximation, these results indicate that the Kaden/Betz model, which assumes conservation of dispersion for each part of the rolled-up vortex, can be improved.

We now discuss rotational energy conservation in the discrete-vortex approximation. For a system of point vortices ($a=0$), it is well known[22] that, besides the conservation of the first and the second moments of vorticity, another vortex invariant is given by

$$W=(1/2\pi)\sum_{i<j}k_i k_j \ln r_{ij} \tag{28}$$

The relation

$$\frac{dw}{dt}=0$$

can easily be confirmed through the equations of motion, Eqs. (27), if $a=0$. W is known as the "Kirchhoff-Routh path function" (see Lin[23]). Lin stated that the function $(-\rho W)$ "may be taken as a measured of the interaction energy" of a vortex system.

It can be shown that $(-\rho W)$ will approach the value of the rotational energy, provided that the vortex field consists of vortex sheets which are represented by a large number of point vortices and that the total vortex strength of rotational field is zero.

As an example, the rotational energy ϵ_R of the field, induced by a vortex sheet with elliptical distribution of circulation, must obey

$$\frac{\epsilon_R}{\rho\Gamma_0^2}=\frac{\pi}{8}=0.393$$

a relation derived from Eqs. (7) and (8). The representation of the vortex sheet by 100 and 200 point vortices of equal strength yields $W/\Gamma_0^2=0.382$ and 0.387, respectively

Although these results are close to the exact value, $-W/\Gamma_0^2=\pi/8$, it can be seen that the structure of the vortex is very different from that found in experiment. The irregular clustering in Fig. 3 is far from axisymmetric, unlike the nearly axisymmetric experimental results. Furthermore, the concentration of vorticity in a small central core, expected from the results of Table 2, is not found in the point-vortex simulation.

The implications of the clustering effect on the conservation of initial W are interesting. Compared with the "linear" arrangement of vortices in the vortex sheet, which implies that every vortex is close to its neighbor, the rollup process spreads the vortices "over an area." Therefore, without clustering of groups of vortices, the relative distance r_{ij} between vortices would increase, thereby yielding a reduced value of W

One way of clustering is the nearly symmetrical concentration of vorticity in vortex cores. The point vortices apparently choose another way—they cluster in a pattern that seems to be chaotic but keeps the value of W invariant. That there is something wrong with the energy can be seen by assuming that the vorticity associated with a point vortex is spread uniformly on a ring with a radius equal to the distance of the vortex from the centroid. The circulation of this distribution as a function of radius is shown in Fig. 6. If we determine the rotational energy, a value of only 85% of the required energy is found.

The commonly used method of reducing the irregularity by introducing an artificial spreading seems to present difficulties. A significant smoothing of the irregular clustering can only be achieved if the core of a single vortex is spread over an area with a mean radius that is very large compared with the distances to neighboring vortices. In this case, the allocation of distributed vorticity to the various discrete vortices seems to be quite arbitrary. Except for this weakness, a strong, symmetric concentration of vorticity, as required for the conservation of rotational energy, cannot be obtained by the method of spreading.

From these simulations it can be concluded that the discrete-vortex approximation cannot be used to compute a maximum tangential speed in agreement with experiment.

However, the Kirchhoff-Routh path function W is an accurate approximation of the rotational energy for homogeneously distributed vorticity This property is used in the next section, where the sensitivity of the simple model, given in the last section, will be checked.

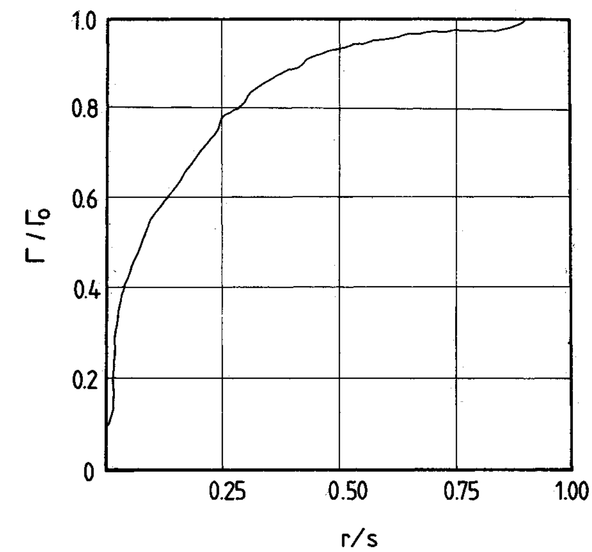

Fig. 6 Averaged vortex circulation vs radius according to point-vortex approximation.

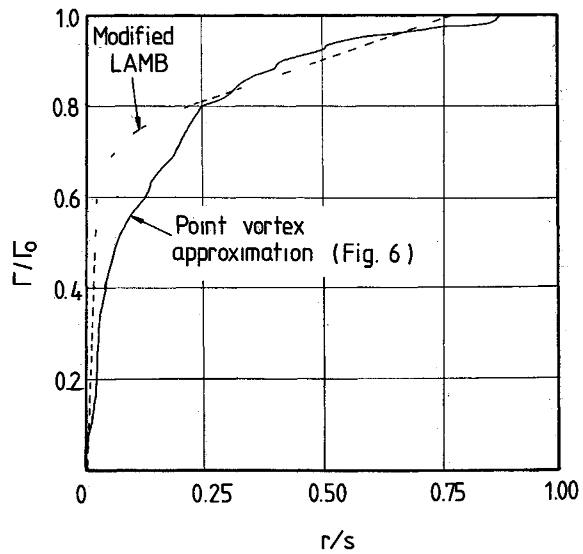

Fig. 7 Modification of the vortex circulation by matching the point-vortex approximation with a modified Lamb core model.

Table 3 Computed vorticity parameters for models 1 4

q	Model	r_C/s	r_K/s	$v_m s/\Gamma_0$
	1	0 078	0 665	1 84
	2	0 115	0 669	1 25
0 9	3	0 095	0 665	1 07
	4	0 132	0 669	1 09
	1	0 057	0.485	2 23
	2	0 082	0 487	1.56
0.8	3	0 067	0 485	1.34
	4	0 093	0 487	1 37
	1	0 033	0 404	3 38
	2	0 053	0 404	2 09
0 7	3	0 044	0 404	1.81
	4	0 061	0 404	1 83

VI. Sensitivity of the Results and Refinement of the Lamb Model

In this section the sensitivity of the results of Sec IV to variations in the distribution of vorticity will be investigated using four models. Both the second moment of vorticity and the rotational energy are conserved in the models, assuming an elliptical loading with circulation Γ_0.

Model 1: The vorticity is concentrated on two concentric circles with radii r_C and r_K The circulation of the inner circle is $q \cdot \Gamma_0 (q<1)$

Model 2: A modified Rankine vortex is assumed that has a homogeneous core ($r \leq r_C$) with circulation $q\Gamma_0$ The remaining vorticity is concentrated on a circle with radius $r_K > r_C$

Model 3: The modified Lamb vortex, discussed in Sec IV.

Model 4: The vorticity in the core region is distributed on a spiral of two revolutions, beginning at r_0 and ending at $r_C = 3r_0$ Again, the circulation around the core is $q\Gamma_0$ and the remaining vorticity is concentrated on a circle of radius r_K

The values of r_C/s, r_K/s, and $v_m s/\Gamma_0$ for these models are listed in Table 3 The parameter q varies from 0 7 to 0.9. In all cases, both the second moment of vorticity and the rotational energy are the same, determined by the elliptical loading The rotational energy was calculated using Eq (28) instead of the approximation given in Sec IV This more exact method leads to larger core radii and smaller values of the tangential speed For example, for the Lamb model with $q = 0\ 7$ we obtain $v_m s/\Gamma_0 = 1\ 81$ instead of the value 3.82 in Table 2

In particular, the interferences between both tip vortices are included in Eq. (28)

With the exception of model 1, which has a singular distribution of vorticity, the tangential speed is not very different for the various models The lower core radius of the Lamb model, together with a smaller value of the circulation, leads to a tangential speed that is qualitatively similar to that of the other models. The values of r_K, necessary for obtaining the required second moment of vorticity, are practically the same for all models Thus the parameters in Table 3 are mainly influenced by the choice of q

For a comparison with experiment, knowledge of q is necessary To compute q we use the averaged circulation function given in Fig. 6 as an approximate description of the vortex structure in the outer region This is reasonable since the point vortex model gives accurate results of the vortical dispersion

Matching one of the core models, such as the Lamb model, to the results of the point vortex simulation leads to the distribution in Fig 7 This is consistent with a value of $q = 0\ 7$ and yields a tangential speed $v_m s/\Gamma_0 = 1\ 45$, which is in good agreement with the experimental results in Fig 2 before the vortex decay begins ($t < 50$s)

In the outer region the distribution of circulation can be approximated by a linear function of r. An analytical model

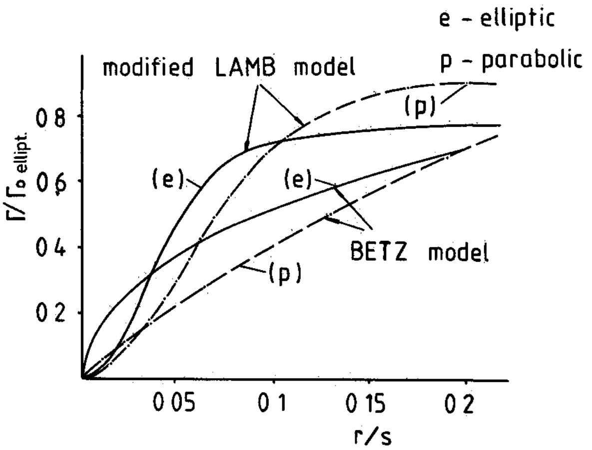

Fig 8 Comparison of the Betz model with the modified Lamb model for elliptic and parabolic wing loading

satisfying the conservation laws can then be given in the form

$$\Gamma/\Gamma_0 = 0\ 7(1 + cr - e^{-\lambda r^2}) \tag{29}$$

This equation uses the Lamb model to match the structure of the vortex core. The parameter c is determined by the known dispersion of vorticity, which depends on the span loading The parameter c and the radius of the vortex r_K can be found by inserting Eq (29) into Eq (16):

$$c = 0\ 136/\sqrt{I/\Gamma_0} = 0\ 136/i \tag{30}$$

and

$$r_K = \sqrt{10I/\Gamma_0} = \sqrt{10}i \tag{31}$$

The parameter λ can then be obtained by a simple iterative procedure, assuming energy conservation, where $\epsilon_R = D_i$ is known from the given span loading.

Figure 8 shows the results of the application of Eq. (29) for elliptic and parabolic wing loadings For comparison, the results of the Betz procedure are also given This figure reveals the importance of satisfying both conservation laws—the conservation of energy as well as vortical disper sion

Conclusion

Neither the known analytical models of tip vortices nor a point vortex simulation can accurately describe experimental results, such as the measured maximum tangential speed As a possible reason for this deficiency of the analytical models the lack of conservation of vortical dispersion and rotational energy was considered The point vortex approximation also fails to conserve the rotational energy because of the nonaxisymmetric, irregular clustering of the point vortices However, this chaotic motion is consistent with the invariance of the Kirchhoff Routh function Only for homogeneously spaced point vortices, however is the value of the Kirchhoff Routh function proportional to the rotational energy of a flowfield with the same averaged vorticity By modifying existing tip vortex models to conserve the vortical dispersion and the rotational energy, it was possible to achieve agreement with experiments The results obtained were not very sensitive to the detailed arrangement of vorticity Finally, a modification of the Lamb vortex model, using a simple analytical formula, is suggested which could describe the radial distribution of circulation for an arbitrary wing loading

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